Problem 1. Coverage problem

We have a finite set $\mathcal{N}$ of players and a finite set $\mathcal{R}$ of locations. The objective is to cover locations with players in order to maximise a social welfare function.

Player i's strategy set is $\mathcal{A}_i = \mathcal{R}$, and hence, each player can choose to cover one location. Let $a = (a_1, \dots, a_N) \in \mathcal{A} = \mathcal{A}_1 \times \dots \mathcal{A}_N$ denote a joint action. Let $\{a\}_r := \{i \in \mathcal{N} \mid r = a_i\} \subseteq \mathcal{N}$ denote the set of players who chose to cover location r and let $W_r : 2^{\mathcal{N}} \to \mathbb{R}_+$ (where $2^{\mathcal{N}}$ is the set of subsets of $\mathcal{N}$) denote the welfare for location r as a function of the players who are assigned to this resource.

As an example, players could be mobile sensors, and the social welfare would be the overall quality of the measurements obtained by the sensors.

The centralised social welfare function $W : \mathcal{A} \to \mathbb{R}$ is then given by

$$W(a) = \sum_{r \in \mathcal{R}} W_r(\{a\}_r). \tag{$\star$}$$

a) Define a utility for player i as $U_i(a) = W_{a_i}(\{a\}_{a_i}) - W_{a_i}(\{a\}_{a_i} \setminus i)$, where $\{a\}_r \setminus i$ is the set of players that are covering location r, except player i. Show that the game is a potential game.
Hint: Consider the social cost function ($\star$) as a candidate potential function.

b) Show that the game in point (a) has a pure strategy Nash equilibrium.

c) Can you derive a bound on the price of anarchy of the game? Justify your answer.

Now consider a special version of this game in which the locations are the R cells = $\{(x_j, y_j)\}_{j=1}^{R}$ of a 2D grid (as in figure below), and consider the players to be mobile sensors. Here, in the figure, we show a discretized grid with 16 locations denoting the cells in the grid. A sensor labeled $\circ$ can move to any of its direct neighbouring locations, labeled $\star$.

4		$\star$		
3	$\star$	$\circ$	$\star$	
2		$\star$		
$y = 1$				
	$x = 1$	2	3	4

The action of each player i is therefore its choice of the grid cell, that is, $a_i = (x_i, y_i) \in \mathcal{R}$. Assume also that each player knows the welfare function W_r for each cell, which denotes the quality of measurements obtained from location r as a function of the number of sensors covering this location r.

d) Write the iterative best-response dynamics for this game. What information does each sensor need to have at each time step to compute its optimal strategy? What strategy profile would the algorithm converge to if any?

e) Now, let us assume that each sensor in location a_i can only determine the number of players that are covering its own location and the immediate neighbouring cells $F(a_i)$, where

$$F(a_i) := \{(x_i + 1, y_i), (x_i, y_i + 1), (x_i - 1, y_i), (x_i, y_i - 1)\}.$$

Also, each sensor at the grid location $a_i = (x_i, y_i)$ can only move to $a_j \in F(a_i)$ or remain in its location (see figure). Assume that sensors are deployed on the grid so that each sensor cannot improve their utility by executing any feasible move. Is this configuration a Nash Equilibrium for the original game, and why? Is the social welfare ($\star$) maximized, and why?

Problem 2. Number of pure strategies in a game in extensive form

Consider a feedback game with T stages, and player 2 having r_t information sets for stage t and k actions per information set. How many pure strategies does player 2 have?

Problem 3. Tic-Tac-Toe

Consider the following famous game:

- **Player 1** draws $\times$ on the board.
- **Player 2** draws $\circ$ on the board.
- The game ends when either Player 1 or Player 2 has put three signs in a row, a column, a diagonal, or when the board is full. See Figure 1.

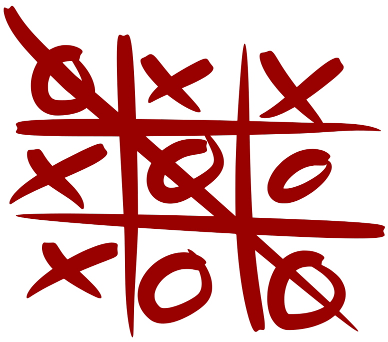

Figure 1: Tic-Tac-Toe

a) Formulate the game in extensive form up to 6 levels (meaning, player 1, player 2, player 1, player 2, player 1, player 2, following only one action at each level).

b) Determine the number of pure strategies of Player 1.